

Due: March 9th (Saturday), 11:59pm, submit to courseworks.

You may use “standard” arguments without a formal proof—either cite from class or add one sentence showing the main idea.

Problem 1: Van Emde Boas insertion

Recall the recursive definition of vEB trees discussed in class. Here’s the pseudo code for the Search operation $\text{PREDECESSOR}(x, S)$ that returns x ’s predecessor within the set S :

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procedure PREDECESSOR( $x, S$ )
  if  $x \leq S.min$  then
    return  $\emptyset$ 
  if  $x > S.max$  then
    return  $S.max$ 
  if  $low(x) > S[high(x)].min$  then
    return  $high(x)\sqrt{u} + \text{PREDECESSOR}(low(x), S[high(x)])$ 
  else
     $i \leftarrow \text{PREDECESSOR}(high(x), S.summary)$ 
    return  $i \cdot \sqrt{|S|} + S[i].max$ 

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Given this code, show that this data structure facilitates dynamic *updates* of inserting keys to S in worst-case $O(\lg w) = O(\lg \lg u)$ time, by completing the pseudo code for $\text{INSERT}(x, S)$.

Problem 2: Fast predecessor search for monotone-interval sets

- (Evenly-spaced intervals) Suppose we are given set of *evenly spaced* numbers $S = x_1 \leq \dots \leq x_n$, i.e. such numbers that there exists a k for which it holds for all $i = 1, \dots, n - 1 : k \leq x_{i+1} - x_i \leq 2k$. Build a linear-space static data structure that answers PREDECESSOR queries on such sequences in $t = O(1)$ time in the RAM model with word size $w = O(\log n)$. (You may use as a black-box the existence of a DICTIONARY data structure, that stores any vector $x \in \Sigma^n$ with d nonzero entries using space $O(d \lg |\Sigma| \lg n)$ and retrieves x_i in constant time $t = O(1)$).
- (Increasing intervals) Do the same for *increasing intervals* sets $x_1 \leq \dots \leq x_n$, in which, for all $i = 1, \dots, n - 2 : x_{i+1} - x_i \leq x_{i+2} - x_{i+1}$. (Hint: Reduce to (1) using fusion trees).

Problem 3: Predecessor applications

Show how to statically solve following problems using PREDECESSOR or SUCCESSOR data structure.

- (1D range reporting) Suppose that for a given list of integers x_1, \dots, x_n you want to answer such queries: given an interval $[a, b]$ return all $x_i \in [a, b]$. $s = O(n)$ space,

$t = O(\log \log u + \text{occ})$ time, where occ is the number of integers in the answer and all $x_i \in \{1, \dots, u\}$.

2. (Subsequence search) Given a string $T \in \Sigma^*$, for any string $P \in \Sigma^*$ determine whether T contains P as a subsequence, i.e. whether there is an increasing sequence of integers $i_1 < \dots < i_{|P|}$, such that $P_j = T_{i_j}$ for all $j \in \{1, \dots, |P|\}$. $s = O(|T| + |\Sigma|)$ space, $t = O(|P| \log \log |T|)$ time.

Problem 4: $O(1)$ predecessor search for near-linear universes

Recall that Van Emde Boas trees solve Predecessor search in a universe of size u with n keys in $t = O(\log \log u)$ time and linear space $s = O(n)$ words, assuming word size $w = O(\log u)$.

We will now show that this running time can be slightly improved for *small* universes $[u]$. Specifically, show that using s words of space (where word size is w bits), we can achieve static search time of $t = O(\log \frac{\log u - \log n}{a})$, where $a := \log(s/n) + \log w$. Conclude that for near-linear universes $u = n \cdot \text{polylog}(n)$, Predecessor search can be done with $s = O(n)$ space, and constant $t = O(1)$ running time (with the standard word-size $w = O(\log n)$).

Hint: Recall that in each round of the vEB recursion, the key-length of elements gets cut by factor 2 (from k to $k/2$, starting at $k = \log u$). First show that we can stop the vEB recursion once k drops below a (indeed, what trivial thing can we do once $k < a$?). Conclude that only $\log(w/a)$ rounds of recursion are needed (rather than $\log w$). Use this simple idea together with simple tabulation, to get constant-time search with linear space for small universes $u = n \cdot \text{polylog}(n)$. If you use this hint without proving it, you will still get partial credit for the problem.