

Due: February 21st (Thursday), 6pm, submit to courseworks.

You may use “standard” arguments without a formal proof—either cite from class or add one sentence showing the main idea.

Problem 1: Working Set implies Entropy

Let T be a binary search tree with a Working Set/Move to Front property, as defined in the second lecture: if $t(x)$ distinct elements were accessed since the last time element x was accessed, then the number of operations performed to access element x will be $O(\log t(x))$ amortized. Prove that this implies that T has entropy property: if access sequence is generated from i.i.d. samples from distribution on $\{1, \dots, n\}$ with probabilities (p_1, \dots, p_n) , then on average T makes $O(\sum_i -p_i \log p_i)$ operations per access.

Problem 2: Splay trees have Working Set property

Prove that splay trees have the Working Set/Move to Front property. Use the Access Lemma proved in class, which states that the amortized search time for key x , with respect to any non-negative weight function $w(x) > 0$, is at most $3(\text{rank}_w(\text{root}) - \text{rank}_w(x)) + 1$. Use the weight function $w(x) = 1/(t(x) + 1)^2$ and the fact that the series $\sum_i 1/i^2$ converges. Note that the weights change after each query, explain how this changes the analysis.

Problem 3: Splay trees insertion analysis

Complete analysis of splay trees: prove that amortized insertion time for splay trees is $O(\log n)$. To do that, look at the inserted node y_0 , let $y_i := \text{parent}(y_{i-1})$, and upper bound the change in potential by $O(\log n)$ when all weights are $w(x) = 1$.